Duration and Convexity Formulas for Odd First Period Bonds
David R Kuipers
Journal of Applied Finance; Spring 2006; 16, 1; ABI/INFORM Global
pg. 82

Duration and Convexity Formulas for Odd First Period Bonds

David R. Kuipers

Some of the most actively traded and liquid instruments in the marketplace are newly issued Treasury notes and bonds. A not inconsiderable proportion of these are issued with irregular first coupon periods. I provide a generalized reduction formula for bond duration and convexity that accommodates instruments with odd first period coupons. Previous duration and convexity reduction formulas for whole-period and fractional-period bonds are contained within the results presented here as special cases. I provide a simple numerical example illustrating the potential calculation errors that can arise if odd first period coupons are ignored. [G12, G13]

US Treasury securities are among the most actively traded and liquid instruments in the marketplace. Many portfolio managers and traders concentrate their hedging activity exclusively in recently auctioned, on-the-run Treasury notes and bonds, instruments which account for the majority of trading volume in the coupon Treasury market (Fleming, 1997). These securities frequently are issued by the government with irregular-length first coupon periods, which results in an *odd first period* payment for the note or bond. As a result, unexpected position losses can arise due to miscalculation, when standard formulas for price, duration, and convexity are used.

I develop a reduction formula for bond duration and convexity that accommodates odd first coupons. Prior results for whole-period and fractional-period bonds, with regular first coupons, are contained within the formula presented here as special cases.[1] While the exposition in the article is developed for Treasury securities in particular, the results are easily adapted to corporate, agency, and municipal bonds priced in the market using alternate day count bases. The only requirement for application of the results derived here, is that a bond make periodic payments that are level after the (possibly odd) first coupon.[2]

David R. Kuipers is an Assistant Professor of Finance at the University of Missouri–Kansas City in Kansas City, MO USA 64110–2499.

I. Background

The US Treasury regularly issues coupon-bearing securities of different maturities to meet the government's borrowing needs. Currently, Treasury coupon issues are non-callable and are priced in the market using the actual/actual day-count convention, with coupon payments paid semiannually relative to the maturity date of the issue. Regular settlement for secondary market transactions is for next-day delivery, and the cash (sometimes called the *dirty*, or *full*) price is at the quoted (*clean*, or *flat*) price plus accrued interest, calculated on a simple interest basis.

[1]Babcock (1984), Bierwag (1987), and Fabozzi and Fabozzi (1995) provide a reduction formula for duration valid on a coupon payment date. Benesh and Celec (1984), Caks, Lane, Greenleaf, and Joules (1985), and Chua (1988) provide duration formulas for fractional-period bonds. Blake and Orszag (1996) outline a convexity formula for whole-period bonds, while Brooks and Livingston (1989) and Nawalkha and Lacey (1991) provide a convexity reduction formula for fractional-period bonds.

[2]There are many alternate ways of defining duration and convexity, depending on assumptions regarding the stochastic behavior of interest rates. Some examples include Fisher and Weil (1971), Cox, Ingersoll, and Ross (1979), Bierwag, Kaufman, and Toevs (1982), Ho (1992), Klaffky, Ma, and Nozari (1992), and Reitano (1992). The formulas in this article are appropriate when using duration and convexity in the manner that they are most traditionally used in the market, which assumes a flat term structure with additive shifts.

The regular payment schedule on a Treasury note or bond suggests the securities should always be issued to the public exactly six months before the first coupon payment date. In practice, this is often not the case, and the first coupon period of the security is something other than six calendar months. There are two reasons for this. First, when a scheduled issue date falls on a weekend or a market holiday, the actual issue date must be adjusted since the security cannot be distributed when the market is closed. Second, at times, the Treasury intentionally dates its securities with irregular issue dates for optimal debt management purposes, to balance the frequency of its refunding auctions with its contemporaneous borrowing needs. Spreading a number of auctions over the calendar year minimizes excess bond supply and dampens bond market volatility, allowing for a smooth distribution of the maturities issued. Scheduling fewer auctions, however, reduces the search costs of auction bidders and creates economies of scale in bidding. As a compromise, the Treasury at times chooses to auction certain notes and bonds together, even though they mature at different calendar dates. In the last 30 years, first-period lengths for some newly issued instruments have been as long as ten months and as short as two months. More than 30% of all coupon notes and bonds auctioned by the Treasury have been issued with an odd first period.

While irregular first coupon periods result either from the Treasury's debt management policy, or idiosyncrasies in the business day calendar, it does not necessarily follow that there will be an odd first coupon payment. This circumstance is directly related to the tax treatment accorded to original issue discount (OID) securities. Consider the effect of an odd first period length on the price of a bond at auction when the first coupon payment is not adjusted. In this case, the adjustment will be incorporated into the auction price, causing short first period bonds to sell at a premium and long first period bonds to sell at a discount. In the latter case, the required price adjustment could potentially be enough to exceed the *de minimus* amount for OID tax treatment. To avoid the problem, the first coupon is normally adjusted so that Treasuries sell at approximately par value at auction, whatever the first period length.[3]

II. General Reduction Formula for Duration and Convexity

In the static price risk analysis of default-free bonds that underlies duration and convexity, we assume that, in a perfect market, the price of a bond is equal to the discounted present value of its future cash flows. Denoting the cash (with-interest) price of a bond as a function of its periodic yield by $B(i)$, a Taylor series expansion of the bond price about a perturbation di results in:

$$dB = -\frac{\partial B}{\partial i}di + \frac{1}{2!}\frac{\partial^2 B}{\partial i^2}(di)^2 + O(i^3) \qquad (1)$$

where $O(i^3)$ represents terms of third-order and higher in i. Dividing through Equation (1) by the bond price and approximating differentials with differences leads to a relation for the percentage price of the bond as a function of its yield:

$$\begin{aligned}\frac{\Delta B}{B} = \%\Delta B &\cong -\left(\frac{1}{B}\right)\frac{\partial B}{\partial i}(\Delta i) + \frac{1}{2}\left(\frac{1}{B}\right)\frac{\partial^2 B}{\partial i^2}(\Delta i)^2 \\ &= -D_{mod}(\Delta i) + \frac{1}{2}\kappa(\Delta i)^2\end{aligned} \qquad (2)$$

where D_{mod} is the modified duration of the bond, and κ is its convexity. At times, it is convenient to calculate alternative coefficients $(1+i)D_{mod}$ and $(1+i)^2\kappa$, which are simply the first- and second-order bond price elasticities with respect to total return $(1+i)$. The first-order elasticity is Macaulay's (1938) duration D_{mac}, also derived by Samuelson (1945), Hicks (1946), and Redington (1952).

As most bonds, including US Treasury securities, make semiannual interest payments, denoting the annualized yield-to-maturity by $y = 2i$ in Equation (2) leads to:

$$\begin{aligned}\%\Delta B &\cong -\left(\frac{D_{mod}}{2}\right)(\Delta y) + \frac{1}{2}\left(\frac{\kappa}{4}\right)(\Delta y)^2 \\ &= -D'_{mod}(\Delta y) + \frac{1}{2}\kappa'(\Delta y)^2 \\ &= \varepsilon_1(\Delta y) + \varepsilon_2(\Delta y)^2\end{aligned} \qquad (3)$$

where both D'_{mod} and κ' are measured in units of years. Equation (3) expresses duration and convexity as commonly defined in the market.[4]

[3]Actual issue prices are slightly different from par, even with the adjustment, because of the discrete nature of Treasury coupon rates. The dating conventions used are well-known and summarized in the Appendix for convenience.

[4]Multiplying Equation (3) through by the bond price, to estimate absolute dollar price differences, results in coefficients often termed the dollar duration and dollar convexity. Some market participants also include the third-order term in Equation (3), which would take the form:

$$\%\Delta B \cong -D'_{mod}(\Delta y) + \frac{1}{2}\kappa'(\Delta y)^2 - \frac{1}{6}\theta'(\Delta y)^3 = \varepsilon_1(\Delta y) + \varepsilon_2(\Delta y)^2 + \varepsilon_3(\Delta y)^3$$

The coefficient θ' is sometimes called a bond's *twist* measure. Though algebraically cumbersome, the reduction formula in Equation (8) and the Exhibit can easily incorporate the twist measure, and is included for the sake of completeness.

To calculate the duration and convexity measures in Equation (3), we begin with the standard formula for the cash price of a bond as a function of its periodic yield:

$$B = \left(\sum_{t=1}^{N} \frac{C_t}{(1+i)^{t-1+F}}\right) + \frac{P}{(1+i)^{N-1+F}} \tag{4}$$

where N is the number of payment dates remaining to maturity, P is the par value of the bond, C is the normal periodic coupon payment, and F is the fraction of the current coupon period remaining. Assuming the bond pays a fixed coupon rate, let TVM be the present value of the cash flow stream in Equation (4) on the most recently scheduled coupon payment date:

$$TVM = \frac{C\left[1-(1+i)^{-N}\right]}{i} + \frac{P}{(1+i)^{N}} \tag{5}$$

TVM is the time value of money bond price found on most simple financial calculators. Let α equal the difference between the next scheduled coupon payment C^*, and a regular coupon payment C, thus allowing for the possibility of odd first coupons:[5]

$$\alpha = C^* - C \tag{6}$$

We can use Equations (5) and (6) to write the price/yield relation in Equation (4) as:

$$B = (1+i)^{1-F}\left\{TVM + \frac{\alpha}{(1+i)}\right\} = B_q + AI \tag{7}$$

where B_q is the quoted bond price, and AI is the accrued interest. Equation (7) is a general price/yield relation for notes and bonds, since α is identically zero at all settlement dates on or after the first coupon date by definition, and is also zero for instruments with normal first coupon period lengths.

To incorporate the possibility of both fractional payment periods and odd first coupons, reduction formulas for bond duration and convexity follow from Equation (2) by taking the appropriate derivatives of Equation (7). After some tedious algebra, Equation (8) holds for the coefficients of any order in Equation (3), where j refers to the order of the coefficient:

$$\varepsilon_j = \left(\frac{1}{j!}\right)\left[\frac{-1}{2(1+i)}\right]^j \left\{aF + \left[\frac{(b/i)TVM + \dfrac{dN(iP-C) - e(1+i)}{i(1+i)^N}}{TVM + \dfrac{\alpha}{(1+i)}}\right]\right\} \tag{8}$$

Hence, ε_1 gives $-D'_{\text{mod}}$, ε_2 gives $\kappa'/2$, and so on. The Macaulay duration measure follows directly from the relation $D_{\text{mac}} = (1+i)D_{\text{mod}}$.[6]

The Exhibit provides the values of a, b, d, and e in Equation (8) for $j = 1$, $j = 2$, and $j = 3$. Note that Equation (8) reduces to familiar forms if α is zero. For example, other than notational differences and algebraic rearrangement, ε_1 with the coefficients in the Exhibit expresses the duration formula in Benesh and Celec (1984), Caks et al. (1985), Chua (1988), and others for fractional-period bonds ($F < 1$), and ε_2 is the convexity formula in Brooks and Livingston (1989) and Nawalkha and Lacey (1991). For whole-period bonds ($F = 1$), ε_1 expresses the duration formula in Babcock (1984) and Bierwag (1987), among others, and ε_2 is the convexity formula in Blake and Orszag (1996). For zero-coupon bonds ($C = 0$), the formula reduces to $F + (N-1)$ for the first-order price elasticity (Macaulay duration) and $F(F+1) + (N+2F)(N-1)$ for the second-order elasticity (return convexity), as expected, with corresponding values of N and $N(N+1)$, respectively, for whole-period zeroes ($F = 1$). Finally, for whole-period par bonds ($iP = C$) Equation (8) reduces to the duration formula in Fabozzi and Fabozzi (1995) and many investment textbooks (e.g., Corrado and Jordan, 2002). The advantage of this reduction formula is that it provides a single, unified approach for the calculation of duration, convexity, and perhaps

[5]The valuation treatment for α and C^* is based on the *Street method*, which uses compound interest for the first cash flow, as opposed to the *Fed method*, which uses simple interest compounding per money market conventions. Standard market practice is to use the Fed method in the when-issued market for coupon Treasury instruments, and the Street method in the secondary market. Procedures for determining the value of F, C^*, and accrued interest AI are provided in the Appendix.

[6]As an alternative to Equation (8), one could begin with the most general bond price/yield relation—with the cash flow payments X_t and time to payment t left completely arbitrary—and rely on a spreadsheet or convenient computing device to find the bond's duration, convexity, and higher-order elasticities. For example, in the case of (Macaulay) duration:

$$B = \sum_t \frac{X_t}{(1+i)^t} \Rightarrow D_{\text{mac}} = \frac{\sum_t \frac{t \cdot X_t}{(1+i)^t}}{B}$$

This approach would be necessary when the payments and/or payment dates are not uniform, as is the case for floating-rate and mortgage-backed bond instruments. For standard, fixed-rate bonds (perhaps with an irregular first payment), Equation (8) offers a closed-form solution that can be calculated with pen and paper—which is desirable to practitioners when access to a computer is inconvenient, and is especially useful in the classroom. In addition, Equation (8) provides a convenient synthesis and generalization of results in earlier research.

Exhibit 1. Constants for Use in Equation (8)

Constants for calculating the coefficients ε_1, ε_2, and ε_3. These coefficients, measured in units of years, are equal to $-D'_{mod}$, $\kappa'/2$, and $-\theta'/6$ for the bond, respectively.

	ε_1	ε_2	ε_3
a	1	$F+1$	$(F+1)(F+2)$
b	1	$(2/i)[1+i(F+1)]$	$3F(F+1)+[6(1+i)/i^2][1+i(F+1)]$
d	1	$2F+N-1$	$F(F-1)+(N+F+1)(N+2F-1)$
e	*P*	$(2/i)[NC+P(1+iF)]$	$3FP(F+1)+(6P/i^2)[1+i(F+1)]+(3NC/i)[N+2F+1]$

higher-order bond price elasticities, while properly handling the case of new-issue notes and bonds with odd first period coupons.[7]

[7]Many software packages and common analysis tools do not correctly handle odd first period bonds. For example, some financial calculators do not accommodate odd first period coupons in the calculation of bond price and yield. Most spreadsheets will do so, but their duration formulas are valid only for fractional-period bonds with normal first coupons, and convexity is not available in any form. Given an initial estimate for the (semiannual) bond yield i_0, users may find a recursion formula based on a Newton-Raphson gradient algorithm useful. Typically, it converges to within a fraction of a basis point of the true yield in just two iterations:

$$i_{t+1} = i_t + \frac{[B/B(i_t)] - 1}{2\varepsilon_1(i_t)}$$

where B is the market price of the bond ($B_q + AI$), $B(i_t)$ is the bond price evaluated at current yield estimate i_t using equation (7), and $\varepsilon_1(i_t)$ is the first-order coefficient in Equation (8) and the Exhibit, evaluated at i_t. For an initial yield estimate i_0, a formula based on the duration of a par bond can be used:

$$i_0 = c + \frac{(1+c) - (B/P)(1+c)^F}{F + (1/c)\left[1-(1+c)^{1-N}\right]}$$

where c is the semi-annual coupon rate expressed in decimal form. This recursion will simultaneously find the correct bond yield, duration, and convexity, including odd first period bonds, subject to the imposed yield accuracy Δi_t. A simple SAS code subroutine implementing the recursion and the equations in the article, will cheerfully be forwarded by the author to interested users upon request.

III. Numerical Example

A numerical example will illustrate the calculation of duration and convexity for odd first period bonds, as well as reveal the typical extent of calculation errors if the odd first coupons are ignored. Assume a hypothetical ten-year Treasury note dated for issue December 1, 2003, with redemption date February 15, 2014, and first coupon date August 15, 2004. This note has a long first coupon period of 8.5 months, not unlike those in many Treasury note auctions in the past. Further suppose the note has an annual coupon rate of 10%, and is quoted on the government interdealer broker exchange at 96% of par for settlement on June 1, 2004.

From the dating conventions provided in the Appendix, it is straightforward to verify that for this hypothetical Treasury note on June 1, 2004, the value of F is 75/182, of G, 107/182 + 76/184, and of α, 380/184. Using these values along with the quoted bond price, Equation (7) can be solved to find the (annualized) bond yield to maturity of 10.659%. At this yield, Equation (8) gives the modified duration as 5.746 years and convexity equal to 47.218 years.

When odd first coupons are ignored, duration and convexity will be overestimated for long first period bonds and underestimated for short first period bonds. The formula provided in Chua (1988), for example, gives a modified duration of 5.860 years using the numbers above, for a percentage calculation error of 1.97% (assuming the correct bond yield is used). The formula in Brooks and Livingston (1989) and Nawalkha and Lacey (1991) gives a convexity result of 48.179 years, for a percentage calculation error of 2.04%. Not surprisingly, the errors are greater if the incorrect bond yield is used. It can be shown that the calculation error for both duration and convexity, if odd first coupons are ignored, is most sensitive to α and the bond coupon rate. For some combinations of high-coupon bonds with long or short first periods, calculation errors can exceed 4%. Though less critical as a source of error, the percentage miscalculation is also an increasing function of bond maturity and a decreasing function of bond price and the fraction F.■

Appendix. Dating Conventions and Odd First Period Coupons for Actual/Actual Bonds

Let Q = settlement date;
A = next scheduled coupon payment date;
B = bond issuance date;
C = (semiannual) coupon payment;
D = regular coupon payment date one period prior to A; and
E = regular coupon payment date two periods prior to A.

The table summarizes the calculation of the variables F (fraction of current coupon period remaining), G (fraction of current coupon period elapsed), and C^* (size of next coupon payment), using the Securities Industry Association (SIA) standards and dating conventions above, for use in Equations (4), (6), (7), and (8). Note that the accrued interest for bonds by market convention is calculated using simple interest as $AI = C \times G$, and that on a coupon payment date, $F = 1$ and $G = 0$ by construction.

	Long First Period	Short First Period	Regular First Period
F	$\min\left(1, \frac{A-Q}{A-D}\right) + \max\left(0, \frac{D-Q}{D-E}\right)$	$\frac{A-Q}{A-D}$	$\frac{A-Q}{A-D}$
G	$\min\left(\frac{Q-B}{D-E}, \frac{D-B}{D-E}\right) + \max\left(0, \frac{Q-D}{A-D}\right)$	$\frac{Q-B}{A-D}$	$\frac{Q-D}{A-D}$
C^*	$C\left(1+\frac{D-B}{D-E}\right)$	$C\left(\frac{A-B}{A-D}\right)$	C

References

Babcock, Guilford C., 1984, "Duration as a Link Between Yield and Value," *Journal of Portfolio Management* 10 (No. 3, Summer), 55–65.

Benesh, Gary A. and Stephen E. Celec, 1984, "A Simplified Approach for Calculating Bond Duration," *Financial Review* 19 (No. 4, November), 394–396.

Bierwag, Gerald O., 1987, *Duration Analysis: Managing Interest Rate Risk*, Cambridge, MA, Ballinger Publishing Co.

Bierwag, Gerald O., Kaufman, George G., and Alden Toevs, 1982, "Single Factor Duration Models in a Discrete General Equilibrium Framework," *Journal of Finance* 37 (No. 2, May), 325–338.

Blake, David and J. Michael Orszag, 1996, "A Closed-Form Formula for Calculating Bond Convexity," *Journal of Fixed Income* 6 (No. 2, June), 88–91.

Brooks, Robert and Miles Livingston, 1989, "A Closed-Form Equation for Bond Convexity," *Financial Analysts Journal* 45 (No. 6, November/December), 78–79.

Caks, John, Lane, William R., Greenleaf, Robert W., and Reginald G. Joules, 1985, "A Simple Formula for Duration," *Journal of Financial Research* 8 (No. 3, Fall), 245–249.

Chua, Jess H., 1988, "A Generalized Formula for Calculating Bond Duration," *Financial Analysts Journal* 44 (No. 5, September/October), 65–67.

Corrado, Charles J. and Bradford D. Jordan, 2002, *Essentials of Investments*, 3rd ed., New York, NY, McGraw Hill-Irwin.

Cox, John C., Ingersoll, Jr., Jonathan E., and Stephen A. Ross, 1979, "Duration and the Measurement of Basis Risk," *Journal of Business* 52 (No. 1, January), 51–61.

Fabozzi, Frank J. and T. Dessa Fabozzi (eds.), 1995, *The Handbook of Fixed Income Securities*, 4th ed., New York, NY, Irwin.

Fisher, Lawrence and Roman L. Weil, 1971, "Coping with the Risk of Interest Rate Fluctuations: Returns to Bondholders from Naive and Optimal Strategies," *Journal of Business* 44 (No. 4, October), 408–431.

Fleming, Michael J., 1997, "The Round-the-Clock Market for US Treasury Securities," *FRBNY Economic Policy Review* 3 (No. 2, July), 9–32.

Hicks, John R., 1946, *Value and Capital*, 2nd ed., Oxford, Clarendon Press.

Ho, Thomas S.Y., 1992, "Key Rate Durations: Measures of Interest Rate Risks," *Journal of Fixed Income* 2 (No. 3, September), 29–44.

Klaffky, Thomas E., Y.Y. Ma, and Ardavan Nozari, 1992, "Managing Yield Curve Exposure: Introducing Reshaping Durations," *Journal of Fixed Income* 2 (No. 4, December), 39–46.

Macaulay, Frederick R., 1938, *Some Theoretical Problems Suggested by the Movements of Interest Rates, Bond Yields, and Stock Prices in the United States Since 1985*, New York, NY, National Bureau of Economic Research.

Nawalkha, Sanjay K. and Nelson J. Lacey, 1991, "Convexity for Bonds with Special Cash Flow Streams," *Financial Analysts Journal* 47 (No. 1, January/February), 80–82.

Redington, F.M., 1952, "Review of the Principles of Life-Office Valuation," *Journal of the Institute of Actuaries* 78, 286–340. Reprinted in: *Pros & Cons of Immunization*, M.L. Liebowitz (ed.), Salomon Brothers, Inc. 1980).

Reitano, Robert R., 1992, "Non-Parallel Yield Curve Shifts and Immunization," *Journal of Portfolio Management* 18 (No. 3, Spring), 36–43.

Samuelson, Paul A., 1945, "The Effect of Interest Rate Increases on the Banking System," *American Economic Review* 35 (No. 2, March), 16–27.