Ranking Asymmetric Auctions

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Abstract

We compare the expected revenue in first- and second-price auctions with asymmetric bidders. We consider a "close to uniform" distributions with identical supports. In contrast to the common conjecture in the literature, we show that in the case of identical supports the expected revenue in second-price auction may exceed that in first-price auction. We also show that asymmetry over lower valuations has a higher negative impact on the expected revenue in first-price auction than in second-price auction. However, asymmetry over high valuations always increases the revenue in first-price auction.

Keywords: asymmetric auctions; ranking auctions; perturbation analysis, revenue equivalence

JEL classification: D44; D72; D82

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1 Introduction

The Revenue Equivalence Theorem (RET) states that the expected revenue of the seller in equilibrium is independent of the auction mechanism under quite general conditions. Since Vickrey (1961) established the revenue equivalence for the classical auction mechanisms (first-price, Dutch, English and second-price auctions) the RET has been considered as one of the most fundamental results in auction theory. Twenty years later, Myerson (1981) and Riley and Samuelson (1981) generalized the RET for symmetric auctions, which are auctions in which the valuations of all the players are drawn from the same distribution function. However, asymmetric auctions where bidders' valuations are drawn from different distributions are not revenue equivalent. For example, the expected revenue in first-price auction can be higher than in second-price auction (see, e.g., Marshall, Meurer, Richard and Stromquist 1994).

Auction theory has dealt mostly with symmetric auctions since in this case an explicit expression for the equilibrium bidding strategies can be obtained. In many realistic situations, however, bidders' valuations are drawn from different distribution functions. Because explicit expressions for asymmetric equilibrium strategies cannot be obtained, except for very simple models, analysis of asymmetric auctions is considerably more complex and after decades of research in auction theory they are still poorly understood. In particular, the problem of ranking the seller's expected revenue between different auction mechanisms under asymmetry remains an open question (see, e.g., Marshall et al. 1994, Maskin and Riley 2000a, Cantillon 2008 and Kirkegaard 2009).

The common conjecture in the literature is that revenue in first-price auction is higher than in second-price auction when the distributions are continuously differentiable over the same support and no minimum bid or participation fee is used. However, we construct a counter example that demonstrates a higher expected revenue in second-price auction. This example is based on the insight we gain through our analysis that asymmetry over higher types increases expected revenue in first-price auction and asymmetry over the lower types decreases revenue in first-price auction more strongly than in second-price auction. In addition, we found that the existence of asymmetry among bidders is what drives revenue ranking, with the shape of the asymmetry being insignificant. Namely, the distributions may demonstrate stochastic domination or may cross over. In both situations, we show that second-price auction may generate a higher revenue than first-price auction.

To compare seller's revenue in asymmetric first- and second-price auctions we start with the benchmark case in which there is an identical uniform distribution for two bidders but then, the distribution function of each player undergoes a mild symmetric change such that the average distribution remains as the original uniform distribution of the benchmark case. This model raises the question of how the seller's expected revenue changes as a result of this weak asymmetry? Since we cannot find the explicit strategies for these equilibrium, there is no exact answer to this question. In situations such as these, where it is difficult or not even possible to obtain exact solutions, much insight can be gained by employing *perturbation analysis.* For example, Fibich, Gavious and Sela (2004) and Fibich and Gavious (2003) analyzed the effect of weak asymmetry on the seller's expected revenue by using perturbation analysis. They found that under the same conditions as those of the classical (i.e., symmetric) auctions, weak asymmetry generates only second order differences in revenues across auction mechanisms. Namely, if ϵ is the level of asymmetry among the distribution functions and $R(\epsilon)$ is the seller's expected revenue in equilibrium, then $R(\epsilon) = R(0) + \epsilon R'(0) + O(\epsilon^2)$, where both $R(0)$ (the seller's expected revenue in the symmetric case) and $R'(0)$ (the leading-order effect of the asymmetry) are independent of the auction mechanism. The value of $R''(0)$, however, does depend on the auction mechanism and thus, the problem of ranking auctions (with respect to expected revenue) remains an open question. Moreover, since the asymmetry among distribution is measured with respect to a single benchmark distribution then, if the benchmark distribution we consider is the average distribution then, $R'(0) = 0$ and the seller's expected revenue becomes $R(\varepsilon) = R(0) + O(\varepsilon^2)$. Lebrun (2009) generalized this result for the case of multidimensional perturbations (i.e., when ε is a vector). Fibich and Gavious (2010) generalized the leading this result for the case of auctions with interdependent valuations. In the present study, we use perturbation analysis and calculate the $O(\varepsilon^2)$ term, which allows us to compare the expected revenues in first- and second-price auctions.

The paper is organized as follows. Section 2 provides some background on asymmetric auctions. Section 3 defines the model and presents illustrative examples. Section 4 presents some preliminary results. Section 5 states the main result together with numerical illustrations of the results. Section 6 concludes. All technical proofs are relegated to appendixes.

2 Background and Examples

The RET theorem guarantees that the expected revenue is identical for all auction mechanisms satisfying bidder risk neutrality, identical distributions over values (symmetry) and anonymity of auction rules with respect to bidders. However, once the assumption of symmetry is violated, revenue varies between different auction mechanisms, in particular first- and second-price auctions. Ranking of expected revenues is not completely understood even for a simple setting with two bidders, the same support for distributions over valuations and continuously differentiable distributions.

The problem of revenue ranking under asymmetry was considered by Maskin and Riley (2000a). They suggested the following mechanism design argument which demonstrates when expected revenue in a second-price auction is higher than in a first-price auction. Assume two bidders with distributions over valuations $F_i(v), i = 1, 2,\ v \in [0, 1]$ and let bidder's i virtual valuations be $J_i(v) = v - \frac{1-F_i(v)}{f_i(v)}, i = 1, 2$ where $f_i(v)$ is bidder's i density. If the equilibrium bid functions in first-price auction $b_i(v), i = 1, 2$ satisfy $b_1(v) < b_2(v)$ and $J_2(v) \leq J_1(v)$ for all $v \in [0, 1]$, then $R^{first} < R^{second}$ where R^{first}, R^{second} are the expected revenues in first- and second-price auctions, respectively. However, the two assumptions $b_1(v) < b_2(v)$ and $J_2(v) \leq J_1(v)$ cannot hold simultaneously if we assume a continuously differentiable distributions over the same support as shown in the following proposition.

Proposition 1 *Assume that distributions are continuously differentiable over the same support* $[0, 1]$ *and* $F_i(0) = 0,\ i = 1, 2$. *Then* $b_1(v) < b_2(v)$ *for every* $v \in [0, 1]$ *implies that*

$J_2(v) > J_1(v)$ *for some* v.

Proof: By Kirkegaard (2009) (Corollary 2), we know that $b_1(v) < b_2(v)$ for every $v \in [0,1]$ implies that $F_1(v)$ first-order stochastically dominates $F_2(v)$, namely, $F_1(v) < F_2(v)$ for every $v \in [0,1]$. Since $F_1(0) = F_2(0) = 0$ and $F_1(1) = F_2(1)$ by stochastic domination it follows that $F_1'(v) = f_1(v) < F_2'(v) = f_2(v)$ for some v. Thus, $(1 - F_1(v)) \geq (1 - F_2(v))$ for every v and $\frac{(1-F_1(v))}{f_1(v)} > \frac{(1-F_2(v))}{f_2(v)}$ for some v. Finally, we conclude that $J_2(v) > J_1(v)$ for some $v \in [0,1]$. $\boxdot$

Thus, the mechanism design arguments given by Maskin and Riley (2000a) cannot give insights regarding the question of revenue ranking when supports are identical for both bidders and distributions are continuously differentiable.[1,2]

Maskin and Riley (2000a) also considered three types of asymmetry between bidders' valuation, one of which demonstrates higher expected revenue in second-price auction. Their specific model considered two distributions over the same support but with a mass point at zero namely, $F_i(0) > 0$. Thus, $F_i(v)$ is not continuous at $v = 0$. In the current study, although we consider continuous distributions, we show that asymmetry over low valuations may reduce expected revenue more strongly in first-price auction than in second-price auction. We will also show that under our setting, asymmetry over high valuations always increases expected revenue in first-price auctions. Marshall et al. (1994)

[1] Further discussion on Maskin and Riley's (2000a) arguments can be found in Gale and Richard (2008).

[2] Kirkegaard (2010) studied a model similar to one of the models in Maskin and Riley (2000a) with two buyers and different supports. He found that the expected revenue in a first-price auction is higher than in a second-price auction.

considered an asymmetric setting with distributions v^α, v^β and demonstrated that expected revenue in a first-price auction is higher than in a second-price auction. A numerical study by Li and Riley (2007) supported the same intuition that under asymmetry and with no reservation price, first-price auction produces higher expected revenue than second-price auction. Similarly, Gayle and Richard (2008) obtained the same result by using a numerical study. Furthermore, also Klemperer (1999) conjectured that it is quite plausible that a first-price auction may be more profitable in expectation than a second-price auction, when all the assumptions for revenue equivalence except symmetry are satisfied.

Studies by Marshall et al. (1994), Li and Riley (2007) and Gayle and Richard (2008) supported this argument. Kirkegaard (2009) offered a new method for analyzing asymmetric first-price auctions which avoids directly tackling the system of differential equations governing equilibrium bids. However, his method does not address revenue ranking.

In the current study we consider an independent private value model of first- and second-price auctions with two bidders and asymmetric distributions. The asymmetry is formed by a symmetric perturbation near uniform distribution. Bidders submit their bids simultaneously and independently and the highest bid wins. The winner obtains the object and pays his bid in first-price auction while in second-price auction he pays second highest bid (the loser bid).

3 The Model

We start with the general setting of asymmetric first- and second-price auctions. Assume two bidders with independent private valuations $v_i, i = 1, 2$. Bidder i's valuation is drawn according to a twice continuously differentiable distribution function $F_i(v)$ over the support $[0, 1]$. Each bidder's valuation is private information and is known only to the bidder. However, the distribution functions $F_i(v), i = 1, 2$ are common knowledge to all the bidders.

First-price auction

In *first-price auction*, bidders submit a sealed bid simultaneously and independently and the highest bidder wins and pays his bid. Bidder i's problem is how to maximize his expected payoff given bidder j's bidding strategy for every valuation v given by

$$\max_b U_i(b; v) = F_j(v_j(b))(v - b), \quad i = 1, 2, j \neq i,$$

where $v_i(b)$ is bidder i's inverse equilibrium bid function.[3]

The first-order condition for equilibrium is given by (see Maskin and Riley 2000a)

$$v_i'(b) = \frac{F_i(v_i(b))}{f_i(v_i(b))} \frac{1}{v_j(b) - b}\ , \ i = 1, 2, j \neq i. \tag{1}$$

In addition, the equilibrium satisfies the initial conditions

[3] $v_i(b)$ exists since the equilibrium bid function $b_i(v)$ exists, is unique and is monotonically increasing (see Riley 1999, Maskin and Riley 2000b, 2003 and Lebrun 1996, 1999, 2006 for existence, uniqueness and monotonicity).

$$v_i(0) = 0,\ \ i = 1,2 \tag{2}$$

and the boundary conditions

$$v_1(\bar{b}) = v_2(\bar{b}) = 1,$$

where $\bar{b}$ is the maximum equilibrium bid and is identical for both bidders.

The seller's expected revenue is given by (see, for example, Fibich and Gavious 2003)

$$R^{first} = E(\max(b_1, b_2)) = \bar{b} - \int_0^{\bar{b}} F_1(v_1(b))F_2(v_2(b))db. \tag{3}$$

In the symmetric case when $F_i = F$, $i = 1, 2$, the bid functions are identical ($b_i(v) = b_{sym}(v)$) and are given by (see Vickery 1961)

$$b_{sym}(v) = v - \frac{1}{F(v)} \int_0^v F(s)ds.$$

Hence, the maximal bid in the symmetric case is given by

$$\bar{b}_{sym} = \ b_{sym}(1) = 1 - \int_0^1 F(s)ds \tag{4}$$

and the expected revenue in the symmetric case is given by (Riley and Samuelson 1981)

$$R^{first}_{sym} = 1 - 2\int_0^1 F(s)ds + \int_0^1 F^2(s)ds. \tag{5}$$

Second-price auction

In *second-price sealed bid auction*, bidders submit a sealed bid simultaneously and independently. The bidder who offers the highest bid wins and pays second highest bid. Submitting bidders' true value (Vikrey 1961) namely, $b(v) = v$, is a weak dominant strategy in either a symmetric or asymmetric setting since asymmetry among bidders does not affect the equilibrium bids in second price auction. The seller's expected revenue under asymmetry is given by (see, for example, Fibich and Gavious 2003)

$$R^{second} = 1 - \int_0^1 \left[F_1(v) + F_2(v) - F_1(v)F_2(v)\right] dv. \tag{6}$$

By the RET (Myerson 1981 and Riley and Samuelson 1981) the sellers' expected revenue in the symmetric model is identical for all auctions. Thus

$$R_{sym}^{first} = R_{sym}^{second}. \tag{7}$$

Hereafter we will use R_{sym} to denote the expected revenue in the symmetric case.

3.1 Examples

In this section, we illustrate some phenomena of revenue ranking with three examples. In first example we show that revenue in first-price auction is higher than in second-price auction as expected by the literature. The revenue for first-price auctions are calculated numerically (see Fibich and Gavious 2003 for further discussion on numerical calculations of revenue in first-price auctions) while the expected revenue in second-price auctions is calculated analytically (there is a closed form expression for revenue in second-price auction, see equation 6 below). Through of this paper we use the notations $R^j, j =$

$first, second$ for the expected revenue in first- and second-price auction and R_{sym} for the expected revenue in the symmetric case with identical distributions which is equal for all auctions according to the RET.

Example 1

Let $F_1(v) = v + 0.2v(1-v)$ and $F_2(v) = v - 0.2v(1-v)$ where $v \in [0,1]$. Solving numerically for the revenue gives $R^{first} = 0.33315$ and calculation gives $R^{second} = 0.332$. We can see that in this particular case revenue in first-price auction is higher than in second-price auction, namely, $\Delta R = R^{first} - R^{second} = 0.00115 > 0$ and the asymmetry is lower for the expected revenue in both, first- and second -price auctions compared to revenue in the symmetric case with $F_1 = F_2 = v$ where $R_{sym} = \frac{1}{3}$. Observe that ΔR is small since, as predicted in Fibich, Gavious and Sela (2003), the difference in revenues between auctions is lower in order comparing to the level of asymmetry (i.e., the difference in revenue is $O(\varepsilon^2)$).

Example 2

As mentioned above, the common conjecture in the literature is that revenue in first-price auction is higher than in second-price auctions at least, for the identical support case as we consider in the current study.

Let $F_1(v) = v + v(1-v)^5$ and $F_2(v) = v - v(1-v)^5$ where $v \in [0,1]$ (see Figures 1 and 2). Then $\Delta R = R^{first} - R^{second} = -0.00045 < 0$ where the revenues are $R^{first} = 0.33171$ and $R^{second} = 0.33216$. Thus, although the the distributions are simple and even demonstrate stochastic dominance $F_2 \succ F_1$, the conjecture about revenue ranking is

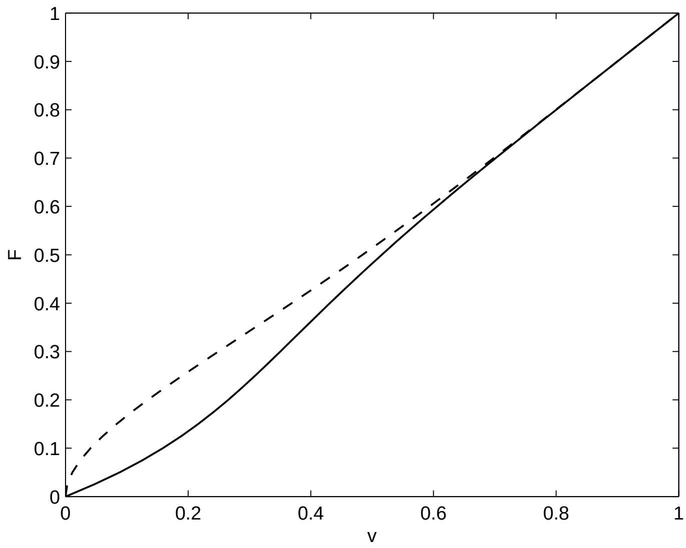


Figure 1: The distributions in Example 2.

incorrect. In the current study we will illuminate this unexpected phenomenon.

As in Example 1, asymmetry lower the expected revenue in first-price auction as well as in second-price auctions comparing to the symmetric case with $F_1(v) = F_2(v) = v$ and $R_{sym} = 1/3$.

Example 3

In examples 1 and 2 we found that expected revenue in the asymmetric first-price auction is lower than the revenue in the symmetric case with uniform distribution. In the following example, we show that asymmetry increases expected revenue in first-price auction while second-price auction is lower under asymmetry (see in Corollary 1 below).

Let $F_1(v) = v + 0.5v^{13}(1 - v)$ and $F_2(v) = v - 0.5v^{13}(1 - v)$ for $v \in [0, 1]$. Then $R^{first} = 0.33334 > \frac{1}{3} = R_{sym}$. Therefore, for this case asymmetry in this case increases expected revenue in first-price auction compared to the symmetric case. Observe that if

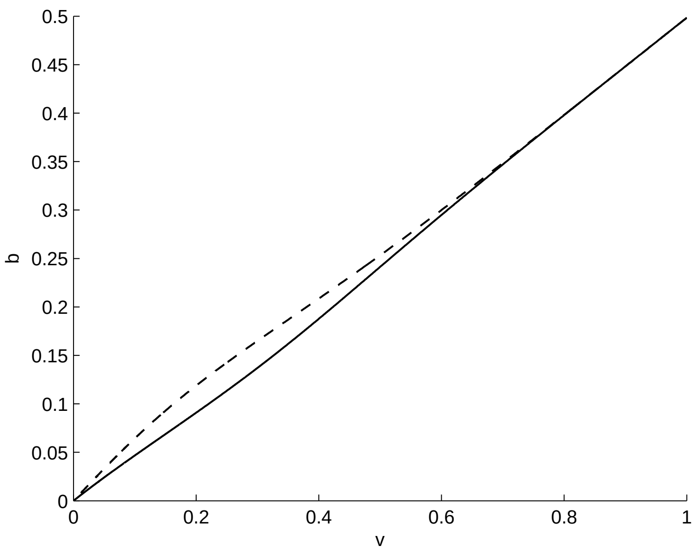
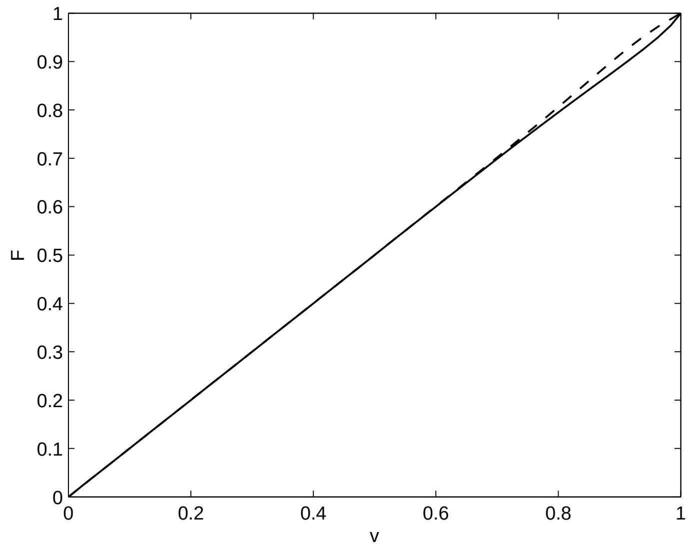
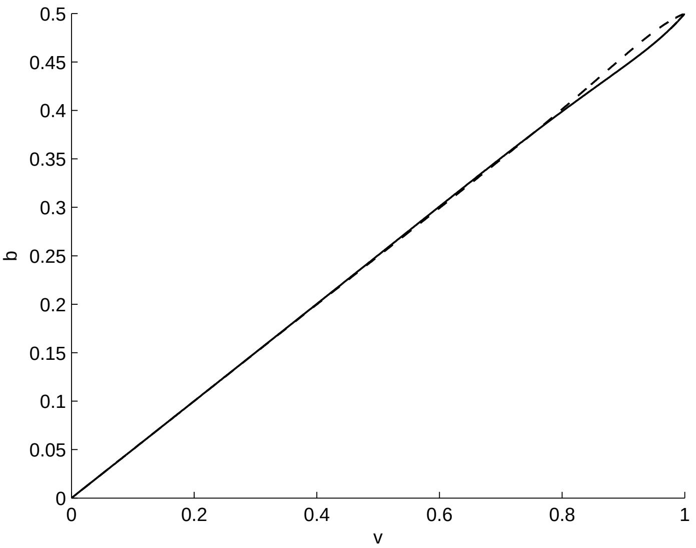


Figure 2: The equilibrium bids in Example 2.

we can find (see Corollary 2) that under some conditions $R^{first} > R_{sym}$ and in addition that $R_{sym} > R^{second}$ then, since by the RET R_{sym} is identical for all auctions, we can conclude that $R^{first} > R^{second}$.



The distributions in Example 3.



The equilibrium bids in Example 3.

We will show that the observation $R^{first} > R_{sym}$ in this example is derived from the specific structure of F_1, F_2 which demonstrate asymmetry over high valuations.

3.2 Assumptions and notations

In this study we consider the following type of distributions

$$F_1(v) = v + \varepsilon H(v), \qquad F_2(v) = v - \varepsilon H(v), \quad v \in [0,1], \quad \varepsilon > 0 \tag{8}$$

where

$$H(0) = H(1) = 0, \quad |H(v)| \leq 1 \tag{9}$$

and ε is small. Assume that the equilibrium bid functions in first-price auction $b_i(v)$ have the following expansion in ε,[4]

$$b_i(v) = b_{sym}(v) + \varepsilon b_{1i}(v) + \varepsilon^2 b_{2i}(v) + O(\varepsilon^3). \quad i = 1,2 \tag{10}$$

where b_{sym} is the symmetric bid function in first-price auction and is given by $b_{sym} = v/2$. Let $\bar{b} = \bar{b}_{sym} + \varepsilon \bar{b}_1 + \varepsilon^2 \bar{b}_2 + O(\varepsilon^3)$ be an expansion of the maximal bid. By Fibich and Gavious (2003), $\bar{b}_1 = 0$ and thus,

$$\bar{b} = \bar{b}_{sym} + \varepsilon^2 \bar{b}_2 + O(\varepsilon^3). \tag{11}$$

Let $v_i(b)$ be the inverse equilibrium bid functions whose expansion in ε is given by

$$v_i(b) = v_{sym}(b) + \varepsilon V_i(b) + \varepsilon^2 U_i(b) + O(\varepsilon^3), \quad i = 1,2 \tag{12}$$

[4]By Lebrun (2009), this assumption is satisfied up to order ε.

where

$$v_{sym}(b) = 2b \tag{13}$$

is the symmetric inverse bid functions in first-price auction. By Fibich and Gavious (2003), we have

$$V_1(b) = -V_2(b) = V(b) \tag{14}$$

where

$$V(b) = 16b^3 \int_{2b}^{1} \frac{H(x)}{x^4} dx - H(2b). \tag{15}$$

4 Expected Revenue and Maximum Bid

In this section we develop an explicit expression for the expected revenue and maximum bid. Observe that for $\varepsilon = 0$ we have $F_1(v) = F_2(v) = v$ and for $\varepsilon > 0$ we have $\frac{1}{2}(F_1(v) + F_2(v)) = v$ (namely, the average distribution is equal to the symmetric distribution for $\varepsilon = 0$). Then, by Fibich, Gavious and Sela (2004), $R^{first} = R_{sym} + O(\varepsilon^2)$ and $R^{second} = R_{sym} + O(\varepsilon^2)$. Thus,

$$R^{first} - R^{second} = O(\varepsilon^2).$$

Since we wish to compare revenues, we have to calculate the ε^2 terms in the expected revenues R^{first} and R^{second}. Note that given the above assumptions, for $\varepsilon = 0$ the model is symmetric with distributions $F_i(v) = v$. Thus, $R_{sym} = \frac{1}{3}$.

4.1 Revenue in the Second-Price Auction

Given the above assumptions we have the exact value for R^{second} as given in the following lemma.

Lemma 1 *Assume that the distributions satisfy (8,9). Then the expected revenue in second-price auction is given by*

$$R^{second} = R_{sym} - \varepsilon^2 \int_0^1 H^2(v)dv. \tag{16}$$

Proof: Substituting (8) in (6) and rearranging yields the result. □

Observe that (16) is exact with no residual. The following is an immediate conclusion of Lemma 1.

Corollary 1

$$R^{second} < R_{sym}.$$

Proof: Directly from (16). □

Thus, asymmetry decreases R^{second} in our model. It is simple to verify that for general distributions which are not limited by our assumptions and $n = 2$, the result still holds. However, we can easily find an example such that for $n > 2$ the result is incorrect, namely, when $R^{second} > R_{sym}$.

4.2 Revenue in the First-Price Auction

In this subsection we derive an expression for R^{first} as well as an expression for the maximal bid $\bar{b}$.

Theorem 1 *Assume that the distributions satisfy (8,9). Then the expected revenue in first-price auction is given by*

$$R^{first} = R_{sym} + \varepsilon^2 \int_0^1 2\left[z^2(w) - \left(w - \frac{9}{8}w^2\right)(z'(w))^2\right] dw + O(\varepsilon^3) \tag{17}$$

where $z(w)$ is given by

$$z(w) = 2w^3 \int_w^1 \frac{H(x)}{x^4} dx. \tag{18}$$

Proof: See Appendix B. □

The following result is a consequence of Theorem 1 and demonstrates how Example 3 was constructed.

Corollary 2 *Assume that $H(v) = 0$ for all $v \in [0, \frac{8}{9}]$ and $H(v) \neq 0$ for some $v \in [\frac{8}{9}, 1]$. Then,*

$$R_{sym} < R^{first}. \tag{19}$$

Proof: For the interval of $[0, \frac{8}{9}]$ $H(v) = 0$. Thus, the integrand in (17) is zero. For the interval $[\frac{8}{9}, 1]$ the integral is strictly positive. ⊡

Thus, asymmetry over high values increases the expected revenue for the seller. Observe that asymmetry over high values generates a higher expected revenue in first-price auction than in second-price auction with no limitation on the shape of the asymmetry. The distributions may cross over a few times or demonstrate stochastic dominance and yet, the existence of asymmetry over the high valuation will suffice for a higher revenue in first-price auction. The following result immediately follows from the above arguments.

Corollary 3 *Assume that* $H(v) = 0$ *for all* $v \in [0, \frac{8}{9}]$ *and* $H(v) \neq 0$ *for some* $v \in [\frac{8}{9}, 1]$. *Then,*

$$R^{second} < R^{first}.$$

Proof: Follows from Corollary 1 and Corollary 2. $\boxdot$

Next, we calculate the maximal bid in first-price auction.

Proposition 2 *The maximum bid is given by*

$$\overline{b} = \overline{b}_{sym} - \varepsilon^2 \int_0^1 \left(z'(w)\right)^2 w dw + O(\varepsilon^3). \tag{20}$$

where $\overline{b}_{sym} = 0.5$.

Proof: See Appendix C. $\square$

The following Corollary follows directly from Proposition 2.

Corollary 4

$$\overline{b}_{sym} > \overline{b}.$$

Proof: Follows immediately from **(20)**.$\square$

In our setting, any asymmetry will decrease the maximal bid,[5] which is quite surprising since, on the one hand we showed that asymmetry over the high valuation increases expected revenue, but, on the other hand, by Corollary 4 we find that equilibrium bids for high valuations are lower than in the symmetric case. Thus, we seem to have a

[5]The result is generally incorrect since we can find other asymmetries which do not satisfy our assumptions. Still, the maximum bid is higher than in the symmetric case.

contradiction, namely that asymmetry near $v = 1$ reduces the maximum bid which implies that also bids for valuations close to 1 are lower than in the symmetric case. Thus, we may expect that revenue will be lower than in the symmetric case. However, by Corollary 2 revenue is higher. The problem can be resolved by the observation that $\bar{b}_{sym} - \bar{b} = O(\varepsilon^2)$ Moreover, it is simple to verify that both bid functions are below the symmetric case (i.e., $b_i(v) < b_{sym}(v), i = 1, 2$) only if v is sufficiently close to 1 namely, $v \in [1 - O(\varepsilon), 1]$. Thus, $\bar{b}_{sym} > \bar{b}$ reduces expected revenue at a magnitude of only $O(\varepsilon^3)$ resolves the contradiction since $R^{first} - R_{sym} = O(\varepsilon^2)$.

5 Comparing Revenues

We now develop an expression for the revenue difference between first and second price auctions. Let

$$\Delta R = R^{first} - R^{second}.$$

Theorem 2

$$\Delta R = \varepsilon^2 \int_0^1 \left[5z^2(w) - 2\left(w - \frac{5w^2}{4}\right)(z'(w))^2 \right] dw + O(\varepsilon^3) \tag{21}$$

where $z(w)$ is given by (18).

Proof: Substituting $v = 2b$, and $z(b)$, $z'(b)$ given by (18) in (16) gives

$$R^{second} = R_{sym} - \varepsilon^2 \int_0^1 \frac{1}{4} \left(3z(w) - z'(w)w\right)^2 dw. \tag{22}$$

Substituting (22) and (17) in ΔR and rearranging terms complete the proof. □

Corollary 5 *Assume that* $H(v) = 0$ *for* $v \in [0, 0.8]$ *and* $H(v) \neq 0$ *for some* $v \in [0.8, 1]$. *Then* $\Delta R \geq 0$.

Proof: Similar to Corollary 2. □

Observe that the interval in Corollary 5 is larger than the interval in Corollary 3, namely, $[8/9, 1] \subset [0.8, 1]$.

It seems that we can use an optimal control technique to find the $z(w)$ that minimizes ΔR in (21) and thus, maximizes the difference between auctions' expected revenues in favor of second-price auction. Obviously, if we can show that the optimal $z(w)$ that minimizes ΔR is identically zero then, we can show that $\Delta R \geq 0$. Thus, expected revenue in first-price auction will always be higher than in second-price auction. However, simple calculation shows that there is no $z(w)$ that minimizes ΔR since we can always define a function $z(w)$ with a lower valuation and sharper derivatives by increasing its fluctuations for lower types.[6] Obviously, a function which demonstrates overly sharp derivatives cannot fit our assumptions since $H'(v)$ should be bounded such that $F_i(v)$ will be monotonic.

The first intuition we learn from Theorem 2 is that if we want to construct an example such that $\Delta R < 0$ we need the integrand in (21) to be small for $v \in [0.8, 1]$. By setting $H(v) = 0$ over $[0.8, 1]$ we can satisfy this requirement. We conclude that asymmetry over the low valuations may lead to $\Delta R < 0$. Moreover, $5z^2(w) - 2\left(w - \frac{5w^2}{4}\right)(z'(w))^2 < 0$ whenever z' is sufficiently large compared to z for example, if we choose z with fluctuations

[6] The second-order condition for optimality is not satisfied for the problem $\min_{z(w)} \Delta R$ which indicates that there is no minimum.

such that z's derivative z' is sharp while the function z is small. To give a formal statement we present the following proposition. Let

$$\Delta(\gamma) = \varepsilon^2 \int_0^\gamma \left[5z^2(w) - 2\left(w - \frac{5w^2}{4}\right)(z'(w))^2 \right] dw$$

and observe that $\Delta(1) = \Delta R + O\left(\varepsilon^3\right)$.

Proposition 3 *If γ is sufficiently small then,*

$$\Delta(\gamma) < 0.$$

Proof: Observe that $z(0) = 0$ and $z'(0) = H'(0)$ and assume that $H'(0) \neq 0$ (this assumption is only for simplicity and can be easily removed). A Taylor expansions around $w = 0$ gives $z(w) = z'(0)w + O\left(w^2\right)$ and $z'(w) = z'(0) + O\left(w\right)$. Substituting the expansions of $z(w)$ and $z'(w)$ in the integrad of $\Delta(\gamma)$ and collecting terms gives

$$\Delta(\gamma) = \varepsilon^2 \int_0^\gamma \left[-2w(z'(0))^2 + O\left(w^2\right) \right] dw$$

which is negative for a sufficienty small w. $\square$

5.1 Numerical illustrations

To demonstrate our findings in the previous section we begin by calculating the expected revenue in first-price auction. We consider the distributions $F_1(v) = v + \varepsilon v(1-v)$, $F_2(v) = v - \varepsilon v(1-v)$. Using (17) and (18) we find that

$$R^{first} = \frac{1}{3} - 0.0047619\varepsilon^2 + O(\varepsilon^3). \tag{23}$$

ε	R^{first}	R^{first}_{approx}	$\frac{R^{first}-R^{first}_{approx}}{R^{first}}\%$
0.1	0.333286	0.3332857	0.00009
0.2	0.3331547	0.3331429	0.00354
0.3	0.332965	0.332905	0.01802
0.4	0.3327616	0.3325714	0.05716
0.5	0.3326082	0.3321429	0.13989

.

Table 1: Comparison between exact expected revenue calculated numerically for the first-price auction and the analytical approximation. The last column is the relative error in percent.

In Table 1 we give the exact revenues calculated numerically for first-price auction The analytical approximation $R^{first}_{approx} = \frac{1}{3} - 0.0047619\varepsilon^2$ given by (23).

As Table 1 shows, the relative error is extremely small. Even for $\varepsilon = 0.5$ it is merely 0.14%. We repeated the same calculation for ε varying from 0.01 to 0.5 by steps of 0.01 and calculate the regression between $\ln\left(R^{first} - R^{first}_{approx}\right)$ and $\ln(\varepsilon)$, and found that $R^2 = 0.99997$ and that the coefficient of $\ln(\varepsilon)$ is 3.983. This coefficient indicates that the structure of the revenue is $R^{first} = R_{sym} + \varepsilon^2 R_2 + O(\varepsilon^4)$. Observe that in the expansion $R^{first} = R_{sym} + \varepsilon R_1 + \varepsilon^2 R_2 + \varepsilon^3 R_3 + \varepsilon^4 R_4 + \cdots$ the coefficients of ε and ε^3 are zero and thus (23) becomes $R^{first} = \frac{1}{3} - 0.0047619\varepsilon^2 + O(\varepsilon^4)$ where $R_4 \neq 0$. The reason for this phenomenon is that R^{first} is an even function in ε since $R^{first}(\varepsilon) = R^{first}(-\varepsilon)$. Thus, its Taylor expansion is only in even powers of ε.

6 Discussion and Further Results

Example 2 demonstrated that in contrast to the common belief the expected revenue in first-price auction is higher than in second-price auction under asymmetry, expected revenue in second-price auction may exceed expected revenue in first-price auction. Using ΔR as given in (21) we provided a counter intuitive example. First, as we show in Corollaries 3 and 5, asymmetry over high valuations will increase revenue in first-price auction. Thus, we looked for distributions that are symmetric for a higher valuation namely, when $H(v)$ is small for v close to 1. Then, we looked for a $z'(w)$ that is sufficiently larger than $z(w)$ for low types (in our setting lower types are below 0.8) what implies that $H(v)$ should have a significant derivative for lower valuations. Example 2 is simple and the distributions demonstrate stochastic dominance. However, stochastic dominance is not a necessary condition for this result. Consider a much more complex example where $F_1(v) = v + 20v(1-v)^3(0.1-v)(0.2-v), F_2(v) = v - 20v(1-v)^3(0.1-v)(0.2-v)$ (observe that these are legitimate distributions i.e., non-decreasing and bounded by 1). Our numerical calculation showed that $\Delta = -0.00107 < 0$. Thus, in order to get $\Delta < 0$ we needed $z(w)$ with low values and a high derivative for low types and a low value and a low derivative for high types. $z(w)$, which demonstrates fluctuations for low types, may strengthen the negative sign of Δ. Thus a lower valuation will lead to a higher revenue in second-price auctions.

A question that arises is whether the finding in this paper is limited to distributions close to uniform distribution. To answer this question we can consider here a model with distributions $F_1 = v^2 + 0.5v(1-v)^5$ and $F_2 = v^2$. A numerical solution demonstrates again, that $\Delta = -0.00012 < 0$. Here the asymmetry structure is not symmetric near v^2 and yet, second-price auction generates higher revenue than first-price auction.

It is also worth noting that our observation that asymmetry over low valuations reduce expected revenue in first-price auction more aggressively than in second-price auction, sheds light on a finding of Maskin and Riley (2000a). They considered a model with two bidders and distributions over the same support. The asymmetry is generated by shifting the probability mass to the lower end point $v = 0$ for one of the bidders. Thus, one of the bidders has value zero with positive probability $F_1(0) > 0$. This setting demonstrates a significant asymmetry for the lower types. Although their model differs from our model in the sense that one of the bidder's distributions is not continuously differentiable at zero, the observation that asymmetry over the lower types may lead to a higher revenue in second-price auction is valid.

References

[1] Cantillon, E., 2008, The Effect of Bidders' Asymmetries on Expected Revenue in Auctions, *Games and Economic Behavior*, 62, 1-25.

[2] Gale, W.R. and Richard, J.F., 2008, Numerical Solutions of Asymmetric, First-Price, Independent Private Values Auctions, *Computational Economics*, 32, 245-278.

[3] Fibich, G. and Gavious, A., 2003, Asymmetric First-Price Auctions - A Perturbation Approach, *Mathematics of Operations Research*, 28, 836-852.

[4] Fibich, G. and Gavious, A., 2010, Asymptotic Revenue Equivalence of Asymmetric Auctions with Interdependent Values, *European Journal of Operational Research*, 206, 496-507.

[5] Fibich, G. Gavious, A., and Sela, A., 2004, Revenue Equivalence in Asymmetric Auctions, *Journal of Economic Theory*, 115, 309-321.

[6] Kirkegaard, R. 2009, Asymmetric First-Price Auctions, *Journal of Economic Theory*, 144, 1617-1635.

[7] Kirkegaard, R. 2010. Ranking Asymmetric Auctions using the Dispersive Order, mimo, Department of Economics,University of Guelph.

[8] Klemperer, P. 1999, Auction Theory: A Guide to the Literature, *Journal of Economic Surveys*, 13 , 227-286.

[9] Lebrun, B., 1996, Existence of an Equilibrium in First Price Auctions, *Journal of Economic Theory,* 7, 421-443.

[10] Lebrun, B., 1999, First Price Auction with N Bidders Case, *International Economic Review*, 40, 1, 125-142.

[11] Lebrun, B., 2006, Uniqueness of the Equilibrium in First-Price Auctions, *Games and Economic Behavior*, 55, 131–151.

[12] Lebrun, B., 2009, Auctions with Almost Homogeneous Bidders, *Journal of Economic Theory*, 144, 1341-1351.

[13] Li, H and Riely, J., 2007, Auction Choice, *International Journal of Industrial Organization*, 25, 1269-1298.

[14] Marshall, R.C., Meurer, M.J., Richard, J-F, and Stromquist, W., 1994, Numerical Analysis of Asymmetric First Price Auctions, *Games and Economic Behavior*, 7, 193-220.

[15] Maskin, E. and Riley, J., 2000a, Asymmetric Auctions, *Review of Economic Studies*, 67, 413-438.

[16] Maskin, E. and Riley, J., 2000b, Equilibrium in Sealed High Bid Auctions, *Review of Economic Studies*, 67, 439-454.

[17] Maskin, E. and Riley, J., 2003, Uniqueness of Equilibrium in Sealed High-Bid Auctions, *Games and Economic Behavior*, 45, 395-409.

[18] Myerson, R.B., 1981, Optimal Auction Design, *Mathematics of Operations Research*, 6, 58-73.

[19] Riley, J.G. and Samuelson, W.F., 1981, Optimal Auctions, *American Economic Review*, 71, 381-392.

[20] Vickrey, W., 1961, Counterspeculation, Auctions, and Competitive Sealed Tenders, *Journal of Finance*, 16, 8-37.

Appendix

A Auxiliary Lemma

Lemma 2

$$F_i(v_i(b)) = 2b + \varepsilon(V_i(b) + H_i(2b) + \varepsilon^2(V_i(b)H_i'(2b) + u_i(b)) + O(\varepsilon^3). \tag{24}$$

$$f_i(v_i(b)) = 1 + \varepsilon H_i'(2b) + \varepsilon^2 V_i(b) H_i''(2b) + O(\varepsilon^3). \tag{25}$$

Proof: Substituting $v_i(b) = v_{sym}(b) + \varepsilon V_i(b) + \varepsilon^2 U_i(b) + O(\varepsilon^3)$ in $F_i(v_i(b))$, expanding the series near $v_{sym}(b)$ and collecting the terms in ε gives

$$\begin{aligned}
F_i(v_i(b)) &= v_i(b) + \varepsilon H_i(v_i(b)) = \left[v_{sym}(b) + \varepsilon V_i(b) + \varepsilon^2 u_i(b) + O(\varepsilon^3)\right] + \\
&\quad + \varepsilon H_i((v_{sym}(b) + \varepsilon V_i(b) + \varepsilon^2 u_i(b) + O(\varepsilon^3)) = \\
&= v_{sym}(b) + \varepsilon\left[V_i(b) + H_i(v_{sym}(b))\right] + \\
&\quad + \varepsilon^2\left[V_i(b)H_i'(v_{sym}(b)) + u_i(b)\right] + O(\varepsilon^3).
\end{aligned}$$

where $\varepsilon H_i((v_{sym}(b) + \varepsilon V_i(b) + \varepsilon^2 u_i(b) + O(\varepsilon^3)) = \varepsilon H_i(v_{sym}(b)) + \varepsilon^2 V_i(b) H_i'(v_{sym}(b)) + O(\varepsilon^3)$. Substituting $v_{sym}(b) = 2b$ (13) completes the proof. A similar calculation proves (25). □

B Proof of Theorem 1

The following notations are used later on in this proof. Let

$$V(b) = y(b) - H(2b) \tag{26}$$

where $V(b)$ is given by (15) and

$$y(b) = 16b^3 \int_{2b}^{1} \frac{H(x)}{x^4} dx. \tag{27}$$

Substituting (11,13,24) in R^{first} (given by 3) we get

$$\begin{aligned} R^{first} &= \bar{b}_{sym} + \varepsilon \bar{b}_1 + \varepsilon_2^2 \bar{b}_2 - \int_0^{\bar{b}_{sym}+\varepsilon\bar{b}_1+\varepsilon^2\bar{b}_2} 4b^2 + \varepsilon^2 \left[2b(V(b)H'(2b) + u_1(b)) - (V(b) + H(2b))^2\right] + \\ &+\varepsilon^2 \left[2b(u_2(b) + V(b)H^{'}(2b))\right] db + O(\varepsilon^3). \end{aligned}$$

Substituting $\bar{b} - \int_0^{\bar{b}} 4b^2 db = R^{first}_{sym} + O(\varepsilon^3)$ given by Lemma 3 below, $\bar{b}_{sym} = 0.5$ (see equation 4) and $\bar{b}_1 = 0$ (see Fibich and Gavious 2003) gives

$$\begin{aligned} R^{first} &= R^{first}_{sym} - \varepsilon^2 \int_0^{0.5} [2b(V(b)H'(2b) + u_1(b)) - (V(b) + H(2b))^2 + \\ &+2b(u_2(b) + V(b)H^{'}(2b)))]db + O(\varepsilon^3) \end{aligned}$$

Denote $u(b) = u_1(b) + u_2(b)$. Rearranging yields

$$\begin{aligned} R^{first} &= R^{first}_{sym} - \varepsilon^2 \int_0^{0.5} 2b[u(b)) + 2V(b)H'(2b)]db \\ &+\varepsilon^2 \int_0^{0.5} (V(b) + H(2b))^2 db + O(\varepsilon^3). \end{aligned}$$

Substituting $2\int_0^{0.5} bu(b)db$ given by Lemma 5 below gives

$$R^{first} = R^{first}_{sym} - \varepsilon^2 \int_0^{0.5} (1-2b)\left[\frac{2H(2b)V(b)+6V^2(b)}{b} + 4H'(2b)V(b)\right] db + \tag{28}$$
$$+\varepsilon^2 \int_0^{0.5} (V(b)+H(2b))^2 db + O(\varepsilon^3).$$

Substituting $V(b)$ given by (26) in (28) we get

$$R^{first} = R^{first}_{sym} - \varepsilon^2 \left\{ \int_0^{0.5} (1-2b)[\frac{6y^2(b) - 10H(2b)y(b) + 4H^2(2b)}{b}]db + \right. \tag{29}$$
$$\left. - \int_0^{0.5} (1-2b)[4H'(2b)y(b) - 4H'(2b)H(2b)] - y^2(b)db \right\} + O(\varepsilon^3)$$

where $y(b)$ is given by (27). Observe that

$$\int_0^{0.5} bH(2b)H'(2b)db = -\frac{1}{4}\int_0^{0.5} H^2(2b)db \tag{30}$$

and

$$\int_0^{0.5} H(2b)H'(2b)db = 0. \tag{31}$$

Substituting (30,31) in (29) gives

$$R^{first} = R^{first}_{sym} - \varepsilon^2 \left\{ \int_0^{0.5} (1-2b)[\frac{6y^2(b) - 10H(2b)y(b) + 4H^2(2b)}{b}]db + \right. \tag{32}$$
$$\left. - \int_0^{0.5} (1-2b)4H'(2b)y(b) - 2H^2(2b) - y^2(b)db \right\} + O(\varepsilon^3).$$

Integration by parts of $\int\limits_0^{0.5}(1-2b)H'(2b)y(b)db$ gives

$$\int\limits_0^{0.5}(1-2b)H'(2b)y(b)db=-\int\limits_0^{0.5}\frac{H(2b)((1-2b)y'(b)-2y(b))}{2}db. \tag{33}$$

Substituting (33) in (32) yields

$$\begin{aligned} R^{first} &= R^{first}_{sym}-\varepsilon^2\left\{\int\limits_0^{0.5}(1-2b)\left[\frac{6y^2(b)-10H(2b)y(b)+4H^2(2b)}{b}\right]db+\right. \\ &\left.-\int\limits_0^{0.5}-2H(2b)((1-2b)y'(b)-2y(b))-2H^2(2b)-y^2(b)db\right\}+O(\varepsilon^3). \end{aligned} \tag{34}$$

Simple calculation shows that

$$y^{'}(b)=48b^2\int\limits_{2b}^{1}\frac{H(x)}{x^4}dx-\frac{2H(2b)}{b}=\frac{3y(b)-2H(2b)}{b} \tag{35}$$

and thus

$$H(2b)=\frac{3y(b)-y'(b)b}{2}. \tag{36}$$

Substituting (35) and (36) in (34) we get

$$\begin{aligned} R^{first} &= R^{first}_{sym}-\varepsilon^2\left\{\int\limits_0^{0.5}(1-2b)(-y(b)y'(b)+\left(y'(b)\right)^2b))db+\right. \\ &\left.-\int\limits_0^{0.5}-(3y(b)-y'(b)b)\cdot((1-2b)y'(b)-2y(b))-\frac{(3y(b)-y'(b)b)^2}{2}-y^2(b)db\right\}+O(\varepsilon^3). \end{aligned} \tag{37}$$

By integration by parts (observe that $y(1)=0$) we have

$$\int_0^{0.5} by'(b)y(b) = -\int_0^{0.5} \frac{y^2(b)}{2} db \tag{38}$$

and

$$\int_0^{0.5} y'(b)y(b) = 0. \tag{39}$$

Substituting (38) and (39) in (37) gives

$$R^{first} = R^{first}_{sym} + \varepsilon^2 \left\{ \int_0^{0.5} 4y^2(b) - (2b - \frac{9}{2}b^2) \left(y'(b)\right)^2 db \right\} + O(\varepsilon^3).$$

Transformation of $w = 2b$ and substituting $z(w) = y(w/2)$ given by (18) yields the result.

□

Lemma 3 $\quad \overline{b} - \int\limits_0^{\overline{b}} 4b^2 db = R^{first}_{sym} + O(\varepsilon^3).$

Proof: Since $\overline{b}_1 = 0$ and $\overline{b}_{sym} = 0.5$ we have $\overline{b} = \overline{b}_{sym} + \varepsilon\overline{b}_1 + \varepsilon^2\overline{b}_2 + O(\varepsilon^3) = 0.5 + \varepsilon^2\overline{b}_2 + O(\varepsilon^3)$. Thus,

$$\begin{aligned} \overline{b} - \int_0^{\overline{b}} 4b^2 db &= \overline{b} - \frac{4}{3}\overline{b}^3 = \frac{1}{2} + \varepsilon^2\overline{b}_2 - \frac{4}{3}\left(\frac{1}{2} + \varepsilon^2\overline{b}_2\right)^3 + O(\varepsilon^3) = \\ &= \frac{1}{3} + O(\varepsilon^3) = R^{first}_{sym} + O(\varepsilon^3). \end{aligned} \tag{40}$$

□

Lemma 4

$$u(b) = \frac{1}{b}\left\{\int_0^b \frac{2H(2s)V(s)+6V^2(s)}{s} - 4H'(2s)V(s) - 2H(2s)H'(2s)ds + \right. \tag{41}$$

$$\left. + \int_0^b 4s\left(H'(2s)\right)^2 - 4sV(s)H''(2s)ds \right\}.$$

where $u(b) = u_1(b) + u_2(b)$.

Proof: The equilibrium equations are given by

$$v_i'(b) = \frac{F_i(v_i(b))}{f_i(v_i(b))}\frac{1}{v_j(b)-b}, \quad i \neq j. \tag{42}$$

Substituting (12,25,24), $v_{sym}(b) = 2b$ and $v_i'(b)$ (obtained by differentiating 12) in (42) we get

$$\begin{aligned} & v_{sym}'(b) + \varepsilon V_i'(b) + \varepsilon^2 u_i'(b) + O(\varepsilon^3) \\ = & \frac{2b + \varepsilon\left[V_i(b) + H_i(2b)\right] + \varepsilon^2\left[V_i(b)H_i'(2b) + u_i(b)\right]}{1 + \varepsilon H_i'(2b) + \varepsilon^2 V_i(b) H_i''(2b) + O(\varepsilon^3)} \times \\ & \times \frac{1}{2b + \varepsilon V_j(b) + \varepsilon^2 u_j(b) - b} + O(\varepsilon^3), \quad i \neq j. \end{aligned} \tag{43}$$

To simplify the calculation we define

$$\begin{aligned} A &= \frac{2b + \varepsilon\left[V_i(b) + H_i(2b)\right] + \varepsilon^2\left[V_i(b)H_i'(2b) + u_i(b)\right]}{1 + \varepsilon H_i'(2b) + \varepsilon^2 V_i(b) H_i''(2b) + O(\varepsilon^3)} + O(\varepsilon^3); \\ B &= \frac{1}{b + \varepsilon V_j(b) + \varepsilon^2 u_j(b)} + O(\varepsilon^3) \end{aligned}$$

When ε is sufficiently small, $\varepsilon H_i'(2b)+\varepsilon^2 V_i(b)H_i''(2b) < 1$ and then, we can use Taylor's expansion $\frac{1}{1+\varepsilon C} = 1-\varepsilon C+(\varepsilon C)^2+O(\varepsilon^3)$ in A and B above. The expansion for A is given by

$$\begin{aligned} A &= \left[2b+\varepsilon\left[V_i(b)+H_i(2b)\right]+\varepsilon^2\left[V_i(b)H_i'(2b)+u_i(b)\right]\right]\times \\ &\quad \left[1-\varepsilon H_i'(2b)-\varepsilon^2 V_i(b)H_i''(2b)+\varepsilon^2 H_i'^2(2b)\right]+O(\varepsilon^3). \end{aligned}$$

Rearranging gives

$$\begin{aligned} A &= 2b+\varepsilon\left[V_i(b)+H_i(2b)-2bH_i'(2b)\right]+ \\ &\quad +\varepsilon^2\left[2bH_i'^2(2b)-2bV_i(b)H_i''(2b)-V_i(b)H_i'(2b)-\right. \\ &\quad \left.-H_i(2b)H_i'(2b)+u_i(b)+V_i(b)H_i'(2b)\right]+O(\varepsilon^3). \end{aligned}$$

Since $0 \leq \frac{\varepsilon V_j(b)+\varepsilon^2 u_j(b)}{b} << 1$ for small ε (it easy to see that $\frac{V_j(b)}{b}$ is bounded for small b) the Taylor expansion of B is given by

$$\begin{aligned} B &= \frac{1}{b+\varepsilon V_j(b)+\varepsilon^2 u_j(b)} = \frac{1}{b}\frac{1}{1+\frac{\varepsilon V_j(b)+\varepsilon^2 u_j(b)}{b}} = \\ &= \frac{1}{b}\left[1-\frac{\varepsilon V_j(b)}{b}+\varepsilon^2\left(-\frac{u_j(b)}{b}+\frac{1}{b^2}(V_j(b)^2\right)\right]+O(\varepsilon^3). \end{aligned}$$

Multiplying A and B we get

$$\begin{aligned}
& A \times B \\
= \; & 2\left[1 - \frac{\varepsilon V_j(b)}{b} + \varepsilon^2\left(-\frac{u_j(b)}{b} + \frac{1}{b^2}(V_j(b)^2\right)\right] + \\
& \frac{\varepsilon\left[V_i(b) + H_i(2b) - 2bH_i'(2b)\right]}{b} \times (1 - \frac{\varepsilon V_j(b)}{b}) + \\
& +\varepsilon^2 \frac{2bH_i'^2(2b) - 2bV_i(b)H_i''(2b) - H_i(2b)H_i'(2b) + u_i(b)}{b} + O(\varepsilon^3).
\end{aligned}$$

Rearranging (43), comparing the $O\left(\varepsilon^2\right)$ order coefficient ($O(1)$ and $O(\varepsilon)$ are known as shown in Section 3) and substituting $V_1 = -V_2$ (14) we get

$$\begin{aligned}
u_i'(b) \;=\; & -(\frac{2u_j(b)}{b}) - (\frac{2H'(2b)V(b)}{b}) - 2V(b)H''(2b) + \\
& +2H'^2(2b) + \frac{V^2(b)}{b^2} + \frac{u_i(b)}{b} \\
& +\frac{H(2b)V(b)}{b^2} - \frac{H(2b)H'(2b)}{b} + \frac{2V^2(b)}{b^2}.
\end{aligned}$$

Let $u(b) = u_1(b) + u_2(b)$ then

$$\begin{aligned}
u' + \frac{u(b)}{b} \;=\; & \frac{2H(2b)V(b) + 6V^2(b)}{b^2} - \frac{4H'(2b)V(b) + 2H(2b)H'(2b)}{b} + \\
& +4H'^2(2b) - 4V(b)H''(2b).
\end{aligned} \tag{44}$$

Equation (44) is a first order ODE, $u' + P(b)u = g(b)$. By (2) the boundary condition is $u(0) = 0$. The solution of (44) is given by :

$$u(b) = \left(m - \int_b^{0.5} g(s) \cdot e^{-\int_s^{0.5} P(x)dx} ds\right) \cdot e^{\int_b^{0.5} P(s)ds}.$$

where $P(s) = \frac{1}{s}$,

$$g(s) = \frac{2H(2b)V(b) + 6V^2(b)}{b^2} - \frac{4H'(2b)V(b) + 2H(2b)H'(2b)}{b} + \tag{45}$$
$$+4H'^2(2b) - 4V(b)H''(2b).$$

and m is a constant. Since

$$\exp\left(\int_b^{0.5} P(s)ds\right) = \frac{1}{2b}.$$

we have

$$u(b) = \frac{m}{2b} - \frac{1}{b}\int_b^{0.5} g(s)sds.$$

By the boundary conditions $u(0) = 0$ it follows that

$$m = 2\int_0^{0.5} g(s)sds.$$

Substituting m in u gives

$$u(b) = \frac{1}{b}\int_0^{b} g(s)sds. \tag{46}$$

Substituting (45) in (46) yields the result. □

Lemma 5

$$2\int_0^{0.5} bu(b) = \int_0^{0.5}\left(\int_0^{b} g(s)sds\right)db$$
$$= \int_0^{0.5}\left\{(1-2b)\left[\frac{2H(2b)V(b)+6V^2(b)}{b}+4H'(2b)V(b)\right]-\right.$$
$$\left.-4bV(b)H'(2b)db\right\}$$

Proof: Let us find $\int_0^{0.5} 2bu(b)db$ where $u(b)$ is given by Lemma 4.

$$\int_0^{0.5} bu(b)db = \int_0^{0.5}\left[\int_0^{b}(\frac{2H(2s)V(s)+6V^2(s)}{s} - 4H'(2s)V(s) - 2H(2s)H'(2s)+\right.$$
$$\left.+4sH'^2(2s) - 4sV(s)H''(2s))ds\right]db.$$

Integrating by parts, multiplying by 2 and using the boundary condition $u(0) = 0$ we get

$$2\int_0^{0.5} bu(b) = \int_0^{0.5}(1-2b)g(b)bdb =$$
$$= \int_0^{0.5}(1-2b)\left[\frac{2H(2b)V(b)+6V^2(b)}{b} - 4H'(2b)V(b) - 2H(2b)H'(2b)+\right.$$
$$\left.+4bH'^2(2b) - 4bV(b)H''(2b)\right]db.$$

Since $V(b)H'(2b) + 2bH''(2b)V(b) = V(b)(H'(2b)b)'$, we have

$$2\int_0^{0.5} bu(b) = \int_0^{0.5}(1-2b)\left[\frac{2H(2b)V(b)+6V^2(b)}{b} - 2H'(2b)V(b)-\right.$$
$$\left.2H(2b)H'(2b) + 4bH'^2(2b) - 2V(b)(H'(2b)b)'\right]db. \tag{47}$$

Substituting $V'(b) = \frac{3V(b)+H(2b)}{b} - 2H'(2b)$ (obtained by differentiating $V(b)$ defined in equation 15) and integrating by parts gives

$$\int_0^{0.5} 2(1-2b)V_i(b)(H_i'(2b)b)'db = \int_0^{0.5} 4V(b)H'(2b)bdb - \tag{48}$$
$$- \int_0^{0.5} (1-2b)(6V(b)H'(2b) + 2H(2b))H'(2b) - 4H'^2(2b)b)db.$$

Substituting of (48) in (47) complete the proof. □

C Proof of Proposition 2

Substituting $\overline{b}_{sym} = 0.5$ in (41) gives

$$u(0.5) = 2\int_0^{0.5} \Big[\frac{2H(2b)V(b) + 6V^2(b)}{b} - 4H'(2b)V(b) - 2H(2b)H'(2b) + $$
$$+4bH'^2(2b) - 4bV(b)H''(2b)]db.$$

Substituting $V(b)H'(2b) + 2bH''(2b)V(b) = V(b)(H'(2b))b)'$ yields

$$u(0.5) = 2\int_0^{0.5} \Big[\frac{2H(2b)V(b) + 6V^2(b)}{b} - 2H'(2b)V(b) - 2H(2b)H'(v_{sym}(b)) + \tag{49}$$
$$+4bH'^2(2b) - 2V(b)(H'(2b)b)']db.$$

Integrating $2\int V(b)(H'(2b)b)'db$ by parts and substituting $V'(b) = \frac{3V(b)+H(2b)}{b} - 2H'(2b)$ gives

$$2\int_0^{0.5} V(b)(H'(2b)b)'db = -\int_0^{0.5} 2\left[\frac{3V(b)+H(2b)}{b} - 2H^{'}(2b)\right](H'(2b)b)db = \tag{50}$$

$$= -\int_0^{0.5} 6V(b)H'(2b) + 2H(2b))H'(2b) - 4H'^2(2b)bdb.$$

Substituting (50) in (49) we get

$$u(0.5) = 2\int_0^{0.5} \frac{2H(2b)V(b)+6V^2(b)}{b} + 4H'(2b)V(b))db. \tag{51}$$

Substituting (26) in (51) gives

$$u(0.5) = 2\int_0^{0.5} \frac{6y^2(b) - 10H(2b)y(b) + 4H^2(2b)}{b} - 4H(2b)H'(2b) + 4y(b)H^{'}(2b)db.$$

Substituting (31) yields

$$u(0.5) = 2\int_0^{0.5} \frac{6y^2(b) - 10H(2b)y(b) + 4H^2(2b)}{b} + 4y(b)H^{'}(2b)db. \tag{52}$$

Substituting the relation (obtained by integration by parts)

$$\int_0^{0.5} H'(2b)y(b)db = -\int_0^{0.5} \frac{H(2b)y'(b)}{2}db$$

in (52) gives

$$u(0.5) = 2\int_0^{0.5} \frac{6y^2(b) - 10H(2b)y(b) + 4H^2(2b)}{b} - 2H(2b)y'(b)db.$$

Substituting (36) gives

$$u(0.5) = 2\int_0^{0.5}(-y(b)y'(b)+(y'(b))^2\,b)) - (3y(b) - y'(b)b)y'(b)db. \quad (53)$$

Substituting (38) and (39) in (53) we get

$$u(0.5) = \int_0^{0.5} 4\,(y'(b))^2\,bdb.$$

Substituting Lemma 6 (below) gives

$$\overline{b}_2 = -\int_0^{0.5} y'^2(b)bdb. \quad (54)$$

Using $\overline{b} = \overline{b}_{sym} + \varepsilon\overline{b}_1 + \varepsilon^2\overline{b}_2 + O(\varepsilon^3)$, $\overline{b}_1 = 0$, $\overline{b}_{sym} = 0.5$ and rearranging in terms of $z(w)$ (see 18) complete the proof. □

Lemma 6

$$u(0.5) = -4\overline{b}_2$$

Proof: Observe that $v_i(\overline{b}) = 1$ for $i = 1,2$. Substituting $\overline{b} = \overline{b}_{sym} + \varepsilon^2\overline{b}_2 + O(\varepsilon^3)$ (recall that $\overline{b}_1 = 0$) and applying a Taylor expansion near $\overline{b}_{sym}$ gives

$$\begin{aligned}
1 &= v_i(\overline{b}) = v_i(\overline{b}_{sym} + \varepsilon^2\overline{b}_2 + O(\varepsilon^3)) = \\
&= v_{sym}(\overline{b}_{sym} + \varepsilon^2\overline{b}_2 + O(\varepsilon^3)) + \varepsilon V_i(\overline{b}_{sym} + \varepsilon^2\overline{b}_2 + O(\varepsilon^3)) + \varepsilon^2 u_i(\overline{b}_{sym} + \varepsilon^2\overline{b}_2 + O(\varepsilon^3)) + O(\varepsilon^3) \\
&= v_{sym}(\overline{b}_{sym}) + \varepsilon^2\overline{b}_2 v'_{sym}(\overline{b}_{sym}) + \varepsilon V_i(\overline{b}_{sym}) + \varepsilon^2 u_i(\overline{b}_{sym}) + O(\varepsilon^3).
\end{aligned}$$

Using $\overline{b}_{sym} = 0.5$, $v_{sym}(\overline{b}_{sym}) = 1$, $v'_{sym} = 2$ and $V_i(0.5) = 0$ gives

$$1 = 1 + 2\varepsilon^2 \overline{b}_2 + \varepsilon^2 u_i(0.5) + O(\varepsilon^3).$$

Equating the ε^2 terms and summing over $i = 1, 2$ gives

$$-4\overline{b}_2 = u_i(0.5) + u_j(0.5) = u(0.5).$$

□